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Periodic Solutions of Coupled Nonlinear Schrödinger Equations in Nonlinear Optics: the Nonresonant Case

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Abstract—We construct periodic solutions to coupled nonlinear one-dimensional Schrödinger equations with periodic boundary conditions in the nonresonant case.

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The purpose of this note is to construct periodic solutions of coupled nonlinear one-dimensional Schrödinger equations with periodic boundary conditions. In particular, these equations are those for the dimensionless envelopes of the electromagnetic fields (u, v) in nonlinear birefringent fibers [1,2]. Precisely, we consider the following equations:

$$\begin{aligned} i\frac{\partial u}{\partial t} &= -\frac{\partial^2 u}{\partial x^2} + q_1(x)\, u + \alpha(x)\, v + g_1(u, \bar{u}, v, \bar{v}, x), \\ i\frac{\partial v}{\partial t} &= -\frac{\partial^2 v}{\partial x^2} + q_2(x)\, v + \alpha(x)\, u + g_2(u, \bar{u}, v, \bar{v}, x) \end{aligned} \tag{1}$$

with $u(x+\pi, t) = u(x)$ and $v(x+\pi, t) = v(x, t)$.

We proceed along the lines of the method recently developed by Craig and Wayne to construct time periodic solutions to the nonlinear wave and Schrödinger equations [3,4]. The results of Craig and Wayne have been recently generalized by Bourgain who has been able to construct quasi-periodic solutions of 1-D nonlinear wave and Schrödinger equations and 2-D nonlinear Schrödinger equations [5,6]. Craig and Wayne follow the Lyapunov-Schmidt argument which consists in splitting the problem in a resonant finite dimensional piece given by the Q-equation and the remainder of the problem, the P-equation, which is infinite dimensional and contains small divisors issues. We suppose that

$$\begin{aligned} g_1(u, \bar{u}, v, \bar{v}, x) &= \partial_{\bar{u}}\, G(u, \bar{u}, v, \bar{v}, x), \\ g_2(u, \bar{u}, v, \bar{v}, x) &= \partial_{\bar{v}}\, G(u, \bar{u}, v, \bar{v}, x), \end{aligned} \tag{2}$$

where $G(u, \bar{u}, v, \bar{v}, x)$ is a real-valued function.

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In nonlinear optics, a typical example is

$$G(u,\bar{u},v,\bar{v},x)=\frac{1}{2}|u|^4+a(x)|u|^2|v|^2+\frac{1}{2}\cdot$$

The function $G(z_1,z_2,z_3,z_4,x)$ is supposed to be periodic in x with period π, analytic in the domain $\{(z_1,z_2,z_3,z_4,x)\in\mathbb{C}^5 \mid |z_i|<1, |\operatorname{Im} x|<\sigma\}$, and

$$G(z_1,z_2,z_3,z_4,x)=\sum_{|\alpha|\geq 4} a_\alpha(x)\, z_1^{\alpha_1} z_2^{\alpha_2} z_3^{\alpha_3} z_4^{\alpha_4}, \tag{3}$$

with $\alpha=(\alpha_1,\alpha_2,\alpha_3,\alpha_4)\in\mathbb{N}^4$ and $|\alpha|=\alpha_1+\alpha_2+\alpha_3+\alpha_4$. Thus, g_1 and g_2 are cubic in $u,\bar{u},v,\bar{v}$.

The set of all functions G satisfying these hypotheses with the topology of uniform convergence is denoted by $\mathcal{A}_\sigma$. We also assume that the potentials $q_1(x)$, $q_2(x)$ and $a(x)$ are periodic in x with periodic π, analytic in the domain $\{x\in\mathbb{C} \mid |\operatorname{Im}\ x|<\sigma\}$. The set of all potentials satisfying these hypotheses again with the topology of uniform convergence is denoted by $\mathcal{Q}_\sigma$.

CASE A. Let us first consider the case where $a\equiv 0$ and $q_1\neq q_2$. Let $(\psi_j(x))_{j\geq 0}$ (respectively, $(\tilde{\psi}_j(x))_{j\geq 0}$) be the normalized eigenfunctions of $-\frac{d^2}{dx^2}+q_1(x)$ (respectively, $-\frac{d^2}{dx^2}+q_2(x)$) with periodic boundary conditions associated with the corresponding eigenvalues $(w_j)_{j\geq 0}$ (respectively, $(\tilde{w}_j)_{j\geq 0}$).

Rescaling the time variable in (1), i.e., setting $\xi=t\Omega$ where Ω is a frequency, we now consider

$$\begin{aligned}-i\Omega\frac{\partial u}{\partial\xi}&=-\frac{\partial^2 u}{\partial x^2}+q_1(x)\,u+g_1(u,\bar{u},v,\bar{v},x),\\ -i\Omega\frac{\partial v}{\partial\xi}&=-\frac{\partial^2 v}{\partial x^2}+q_1(x)\,v+g_2(u,\bar{u},v,\bar{v},x).\end{aligned} \tag{4}$$

Set

$$\Phi_{2j}(x)=\begin{pmatrix}\psi_j(x)\\ 0\end{pmatrix},\qquad \Phi_{2j+1}(x)=\begin{pmatrix}0\\ \tilde{\psi}_j(x)\end{pmatrix}.$$

Looking for a periodic solution $\begin{pmatrix}u(x,\xi)\\ v(x,\xi)\end{pmatrix}$ to (4) we expand it in an eigenfunction expansion

$$\begin{pmatrix}u(x,\xi)\\ v(x,\xi)\end{pmatrix}=\sum_{(j,k)\in\mathbb{N}\times\mathbb{Z}}\hat{u}(j,k)\,\Phi_j(x)\,e^{-ik\xi}. \tag{5}$$

Insert (5) into (4) to get the nonlinear equation

$$W(\hat{u})(j,k)+V(\Omega)\,\hat{u}(j,k)=0,\qquad j\in\mathbb{N},\quad k\in\mathbb{Z} \tag{6}$$

with

$$\begin{aligned}V(\Omega)\,\hat{u}(2j,k)&=(w_j-k\Omega)\,\hat{u}(2j,k),\\ V(\Omega)\,\hat{u}(2j+1,k)&=(\tilde{w}_j-k\Omega)\,\hat{u}(2j+1,k)\end{aligned}$$

and

$$W(\hat{u})(j,k)=\int_0^\pi\int_0^{2\pi}\left\langle\begin{pmatrix}g_1(u,\bar{u},v,\bar{v},x)\\ g_2(u,\bar{u},v,\bar{v},x)\end{pmatrix},\Phi_j(x)\,e^{-ik\xi}\right\rangle_{\mathbb{C}^2}dx\,d\xi,$$

with $W(T_\theta\hat{u})=T_\theta W(\hat{u})$, where $(T_\theta\hat{v})\,(j,k)=e^{ik\theta}\hat{v}(j,k)$.

Equation (6) is split into a finite dimensional and an infinite dimensional part according to the Lyapunov-Schmidt procedure.

Fix $j_0\in\mathbb{N}$ and set

$$N=\{(j,k)\in\mathbb{N}\times\mathbb{Z}\mid V(w_{j_0})\,(j,k)=0\}. \tag{7}$$

Generically, for $q_1, q_2 \in \mathcal{Q}_\sigma$, and $G \in \mathcal{A}_\sigma$, $\dim N = 1$ [3] and from now on, we suppose that $N = \{(2j_0, 1)\}$. We could have fixed $\tilde{w}_{j_0}$ as well.

Given N, let Q be the orthogonal projection onto $\ell^2(N) \subset \ell^2(\mathbb{N} \times \mathbb{Z})$ and set $P = Q^\perp$. Equation (6) is equivalent to

$$P\,W(\hat{u})\,(j,k) + P\,V(\Omega)\,\hat{u}(j,k) = 0, \tag{8}$$

$$Q\,W(\hat{u})\,(j,k) + Q\,V(\Omega)\,\hat{u}(j,k) = 0. \tag{9}$$

Equation (8) is solved by using a Nash-Moser implicit function theorem and (9) by using bifurcation theory. In order to apply a Nash-Moser procedure, it is necessary to introduce a family of Hilbert subspaces of $\ell^2(\mathbb{N} \times \mathbb{Z})$, i.e., the spaces denoted by $H_{m,s}$, $m \geq 0$, $s \geq 0$ and defined in terms of the $\hat{u}(j,k)$'s as in [4]. Futhermore, we need to satisfy nonresonance conditions. Thus, we suppose that the two sequences $(w_j)_{\substack{j\geq 0\\ j\neq j_0}}$ and $(\tilde{w}_j)_{j\geq 0}$ are (d_0, L_0) nonresonant with $\tilde{w}_{j_0}$, i.e., that, for every $(j,k) \in B(L_0)\backslash N$, we have

$$|kw_{j_0} - w_j| > d_0, \qquad |kw_{j_0} - \tilde{w}_j| > d_0, \tag{10}$$

with L_0 sufficiently large and d_0 depending in L_0 in an appropriate way (cf. [4]). Here, $B(L_0) = \{(j,k) \mid j + |k| \leq L_0\}$ and we suppose that $N \subset B(L_0)$.

Solutions to (8) are constructed by perturbing the solution $\varphi(p) = p\,\delta_{\{(j,k),(j_0,1)\}}$ of the linear part with $p \in \mathbb{C}$. A major part is devoted to the analysis of the linearized operator $PV(\Omega) + DPW(\hat{u})$. Inversion of linear operators proceeds along the lines of the method developed by Fröhlich and Spencer [7]. The only condition that the nonlinear part of (9) must satisfy is the so-called twist condition

$$\frac{\partial^3}{\partial r^3}\,W(\varphi(r))\Big|_{r=0}(2j_0, 1) \neq 0. \tag{11}$$

This condition is used to apply the Morse Lemma.

According to (3), the twist condition is

$$\int_0^\pi a_{2,2,0,0}(x)|\psi_{j_0}(x)|^4\,dx \neq 0.$$

Fixing $\tilde{w}_{j_0}$, we obtain

$$\int_0^\pi a_{0,0,2,2}(x)\left|\tilde{\psi}_{j_0}(x)\right|^4\,dx \neq 0.$$

We finally get the following theorem.

THEOREM 1. *Consider (1) with $\alpha = 0$, $q_1 \neq q_2$ and periodic boundary conditions. Consider w_{j_0}. There exists an open dense set $\mathcal{D}_{j_0} \subset \mathcal{A}_\sigma \times \mathcal{Q}_\sigma \times \mathcal{Q}_\sigma$ such that, if $(G, q_1, q_2) \in \mathcal{D}_{j_0}$, equation (1) has time periodic solutions close to $\Phi_{2j_0}(x)\,e^{iw_{j_0}t}$.*

More precisely, there exists $r_0 > 0$, a C^3 function $\Omega(r)$ defined on $[0, r_0)$ and a Cantor set $C \subset [0, r_0)$ of positive measure, such that, if $r \in C$, there exist time periodic solutions $u(x,t;r)$ and $v(x,t;r)$ of (1) with frequency $\Omega(r)$ such that

$$\left|u(x,t;r) - r\,\psi_{j_0}(x)\,e^{i\Omega(r)t}\right| \leq C\,r^2, \qquad |v(x,t;r)| \leq C\,r^2,$$

and

$$|\Omega(r) - w_{j_0}| \leq C\,r^2.$$

REMARKS.

(i) The same statement is true if we consider $\tilde{w}_{j_0}$ instead of w_{j_0} with obvious changes.
(ii) $\{\Omega(r) \mid r \in C\}$ is also a Cantor set and $0 \in C$.
(iii) Twist conditions are partly used to show that the measure of C (respectively, $\tilde{C}$) is greater than $r_0/2$.

CASE B. We now consider (1) in the case where $\alpha \neq 0$ and $q_1 = q_2 = q$ with $\alpha, q \in \mathcal{Q}_\sigma$. We then have the following system:

$$\begin{aligned} i\frac{\partial u}{\partial t} + \frac{\partial^2 u}{\partial x^2} &= q(x)\,u + \alpha(x)\,v + g_1(u,\bar{u},v,\bar{v},x), \\ i\frac{\partial v}{\partial t} + \frac{\partial^2 v}{\partial x^2} &= q(x)\,v + \alpha(x)\,u + g_2(u,\bar{u},v,\bar{v},x). \end{aligned} \tag{12}$$

Set

$$U(x,t) = u(x,t) + v(x,t) \quad \text{and} \quad V(x,t) = u(x,t) - v(x,t).$$

Then, (12) is equivalent to

$$\begin{aligned} i\frac{\partial U}{\partial t} + \frac{\partial^2 U}{\partial x^2} &= (q(x) + \alpha(x))\,U + \tilde{g}_1(U,\bar{U},V,\bar{V},x), \\ i\frac{\partial V}{\partial t} + \frac{\partial^2 V}{\partial x^2} &= (q(x) - \alpha(x))\,V + \tilde{g}_2(U,\bar{U},V,\bar{V},x), \end{aligned} \tag{13}$$

with $\tilde{g}_1 = \partial_{\bar{U}}\,\tilde{G}$, $\tilde{g}_2 = \partial_{\bar{V}}\,\tilde{G}$ and

$$\tilde{G}(U,\bar{U},V,\bar{V},x) = G(u,\bar{u},v,\bar{v},x).$$

Therefore, we can now apply Theorem 1.

Let $(\varphi_j(x))_{j\geq 0}$ (respectively, $(\tilde{\varphi}_j(x))_{j\geq 0}$) be the normalized eigenfunctions of $-\frac{d^2}{dx^2} + q + \alpha$ (respectively, $-\frac{d^2}{dx^2} + q - \alpha$) with periodic boundary conditions associated with the corresponding eigenvalues $(\lambda_j)_{j\geq 0}$ (respectively, $(\tilde{\lambda}_j)_{j\geq 0}$).

Consider λ_{j_0}. Generically, for $(G,q,\alpha) \in \mathcal{A}_\sigma \times \mathcal{Q}_\sigma \times \mathcal{Q}_\sigma$, $\dim N = \{(j_0, 1)\}$. Suppose that $(\lambda_j)_{\substack{j\neq j_0\\ j\geq 0}}$ and $(\tilde{\lambda}_j)_{j\geq 0}$ satisfy nonresonance conditions with respect to λ_{j_0} as in part A. We then have the following theorem.

THEOREM 2. *Consider (12) with periodic boundary conditions. Consider λ_{j_0}. There exists an open dense set $\mathcal{F}_{j_0} \subset \mathcal{A}_\sigma \times \mathcal{Q}_\sigma \times \mathcal{Q}_\sigma$ such that, if $(G,\alpha,q) \in \mathcal{F}_{j_0}$, then (12) has time periodic solutions close to $\begin{pmatrix} \varphi_{j_0}(x)\, e^{i\lambda_{j_0} t} \\ \varphi_{j_0}(x)\, e^{i\lambda_{j_0} t}\end{pmatrix}$. Thus, there exists $\rho_0 > 0$, a C^3 function $F(\rho)$ defined on $[0,\rho_0)$ and a Cantor set $K \subset [0,\rho_0)$ of positive measure, such that, if $\rho \in K$, there exist time periodic solutions $u(x,t;\rho)$ and $v(x,t;\rho)$ of (12) with frequency $F(\rho)$ such that*

$$\begin{aligned} \left|u(x,t;\rho) - \rho\,\varphi_{j_0}(x)\, e^{iF(\rho)t}\right| &\leq C\,\rho^2, \\ \left|v(x,t;\rho) - \rho\,\varphi_{j_0}(x)\, e^{iF(\rho)t}\right| &\leq C\,\rho^2, \\ |F(\rho) - \lambda_{j_0}| &\leq C\,\rho^2. \end{aligned}$$

Remarks following Theorem 1 are still true for Theorem 2. If we consider $\tilde{\lambda}_{j_0}$ instead of λ_{j_0}, time periodic solutions to (12) are close to $\begin{pmatrix} \varphi_{j_0}(x) \\ -\varphi_{j_0}(x)\end{pmatrix} e^{i\tilde{\lambda}_{j_0} t}$.

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